

## Decay of particles above threshold in the Ising field theory with magnetic field

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### Abstract

The two-dimensional scaling Ising model in a magnetic field at critical temperature is integrable and possesses eight stable particles  $A_i$  ( $i = 1, \dots, 8$ ) with different masses. The heaviest five lie above threshold and owe their stability to integrability. We use form factor perturbation theory to compute the decay widths of the first two particles above threshold when integrability is broken by a small deviation from the critical temperature. The lifetime ratio  $t_4/t_5$  is found to be 0.233; the particle  $A_5$  decays at 47% in the channel  $A_1 A_1$  and for the remaining fraction in the channel  $A_1 A_2$ . The increase of the lifetime with the mass, a feature which can be expected in two dimensions from phase space considerations, is in this model further enhanced by the dynamics.

# 1 Spectrum and analyticity in Ising field theory

Quantum field theory in  $1 + 1$  dimensions allows for a subclass of integrable cases. While integrability is hardly detectable at the level of correlation functions, it has very visible implications at the scattering level. Indeed, the presence of infinitely many conserved quantities forces the  $S$ -matrix to be completely elastic and factorized [1]. These properties, in turn, allow for a feature of the particle spectrum which is peculiar to integrable quantum field theories: the existence of particles with mass higher than the lowest threshold which are stable even in the absence of an internal symmetry which would prevent them from decaying. Once integrability is broken by an arbitrarily small perturbation, the stability of these particles would contradict though the analyticity requirements on the  $S$ -matrix, so that they are forced to develop a finite lifetime. This phenomenon already appears in the field theory describing the simplest universality class of critical behavior, namely that of the Ising model in a magnetic field [2]. The quantitative study of the decay processes following the breaking of integrability in Ising field theory is the subject of this paper.

The scaling Ising model in two dimensions is described by the action

$$\mathcal{A} = \mathcal{A}_0 - \tau \int d^2x \varepsilon(x) - h \int d^2x \sigma(x), \quad (1)$$

in which the critical point action  $\mathcal{A}_0$  is perturbed by the energy operator  $\varepsilon(x)$ , with scaling dimension  $X_\varepsilon = 1$ , and by the spin operator  $\sigma(x)$ , with scaling dimension  $X_\sigma = 1/8$ . The coupling constants<sup>1</sup>

$$\begin{aligned} \tau &\sim M^{2-X_\varepsilon} = M, \\ h &\sim M^{2-X_\sigma} = M^{15/8} \end{aligned}$$

measure the deviation from critical temperature and the magnetic field, respectively, and enter the dimensionless parameter

$$\eta = \frac{\tau}{|h|^{8/15}} \quad (2)$$

which can be used to label the renormalization group trajectories flowing out of the critical point.

It is well known that the action (1) with  $h = 0$  describes free excitations with fermionic statistics. These are ordinary particles in the high-temperature phase  $\tau > 0$  ( $\eta = +\infty$ ), and topological excitations (kinks) interpolating between the two degenerate vacua of the  $\tau < 0$  phase ( $\eta = -\infty$ ) in which the spin reversal symmetry is spontaneously broken. The evolution of the particle spectrum at  $h \neq 0$  was first discussed in [3]. In the limit  $\eta \rightarrow -\infty$  the particle spectrum exhibits infinitely many discrete levels generated by the confinement of the kink-antikink pairs induced by an arbitrarily small magnetic field removing the degeneracy among the two vacua of the low-temperature phase. Hence, the spectrum in this limit is reproduced by

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<sup>1</sup>The mass scale  $M$  is the inverse of the correlation length.

the non-relativistic result for a linear confining potential<sup>2</sup>

$$m_n \simeq 2m + \frac{(2vh)^{2/3} z_n}{m^{1/3}}, \quad (3)$$

where  $m$  is the mass of the kinks,  $v$  the spontaneous magnetization and  $z_n$ ,  $n = 1, 2, \dots$ , are positive numbers determined by the zeroes of the Airy function,  $\text{Ai}(-z_n) = 0$ . Of course, the particles with masses  $m_n$  larger than twice the lightest mass  $m_1$  are unstable. It was argued in [3] that the number of stable particles decreases as  $\eta$  is increased, until a single asymptotic particle is left in the spectrum at  $\eta = +\infty$ . This general pattern has been confirmed by numerical investigation of the spectrum of the field theory (1) [4, 5]. In particular, it has been found that the second and third lightest particles become unstable at

$$\eta_2 = 0.333(7), \quad \eta_3 \simeq 0.022, \quad (4)$$

respectively [5]. These values of  $\eta$ , as well as the other normalization-dependent numbers we quote in the following, refer to the normalization of the operators in which

$$\begin{aligned} \langle \sigma(x) \sigma(0) \rangle &\rightarrow |x|^{-1/4} \\ \langle \varepsilon(x) \varepsilon(0) \rangle &\rightarrow |x|^{-2} \end{aligned} \quad (5)$$

as  $|x| \rightarrow 0$ . Corrections to the mass  $m_1$  for  $\eta \rightarrow +\infty$  as well as to the spectrum (3) have also been computed [5, 6].

A. Zamolodchikov discovered at the end of the eighties that the Ising field theory (1) is integrable for  $\tau = 0$  and computed the exact  $S$ -matrix through the bootstrap method [2]. He found, in particular, that the spectrum of the theory on the magnetic axis contains eight stable particles  $A_a$  ( $a = 1, \dots, 8$ ) with masses<sup>3</sup>

$$\begin{aligned} m_1 &= (4.40490857..) |h|^{8/15} \\ m_2 &= 2m_1 \cos \frac{\pi}{5} = (1.6180339887..) m_1 \\ m_3 &= 2m_1 \cos \frac{\pi}{30} = (1.9890437907..) m_1 \\ m_4 &= 2m_2 \cos \frac{7\pi}{30} = (2.4048671724..) m_1 \\ m_5 &= 2m_2 \cos \frac{2\pi}{15} = (2.9562952015..) m_1 \\ m_6 &= 2m_2 \cos \frac{\pi}{30} = (3.2183404585..) m_1 \\ m_7 &= 4m_2 \cos \frac{\pi}{5} \cos \frac{7\pi}{30} = (3.8911568233..) m_1 \\ m_8 &= 4m_2 \cos \frac{\pi}{5} \cos \frac{2\pi}{15} = (4.7833861168..) m_1. \end{aligned}$$

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<sup>2</sup>The spectrum (3) was originally obtained in [3] through the study of the analytic structure in momentum space of the spin-spin correlation function for small magnetic field.

<sup>3</sup>The constant relating  $m_1$  to the magnetic field was determined in [7].

Notice that the last five masses all lie above the lowest two-particle threshold  $2m_1$ . Since the Ising model in a magnetic field possesses no internal symmetries (so that all particles are completely neutral), the decay processes  $A_a \rightarrow A_1 A_1$ ,  $a = 4, \dots, 8$ , are allowed both by kinematics and by symmetry. The stability of the particles above threshold, however, does not violate any requirement of the  $S$ -matrix theory as long as integrability is present.

To see this, consider for example the scattering amplitudes of  $A_1$  with itself and with  $A_2$ . They read<sup>4</sup> [2]

$$S_{11}(\theta) = t_{2/3}(\theta)t_{2/5}(\theta)t_{1/15}(\theta) \quad (6)$$

$$S_{12}(\theta) = t_{4/5}(\theta)t_{3/5}(\theta)t_{7/15}(\theta)t_{4/15}(\theta), \quad (7)$$

with

$$t_\alpha(\theta) = \frac{\tanh \frac{1}{2}(\theta + i\pi\alpha)}{\tanh \frac{1}{2}(\theta - i\pi\alpha)}. \quad (8)$$

The simple poles at  $\theta = iu_{ab}^c$  in the amplitude  $S_{ab}(\theta)$  indicate that the particle  $A_c$  with mass square

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos u_{ab}^c \quad (9)$$

appears as a bound state in the scattering channel  $A_a A_b$ . Hence we see that the scattering channel  $A_1 A_1$  produces the first three particles as bound states, while the channel  $A_1 A_2$  produces the first four. Figures 1.a and 1.b show the corresponding analytic structure for the two amplitudes in the complex plane of the Mandelstam variable  $s$  (square of the center of mass energy). As required by the  $S$ -matrix theory, all the poles corresponding to stable bound states lie on the real axis below the lowest unitarity threshold in the given scattering channel. This is true for all the amplitudes  $S_{ab}$ ,  $a, b = 1, \dots, 8$ , of the Ising field theory at  $\tau = 0$  (see e.g. [8] for the full list of amplitudes). When integrability is broken (i.e. as soon as we move away from  $\tau = 0$ ), however, the inelastic channels and the associated unitarity cuts open up. In particular, the process  $A_1 A_2 \rightarrow A_1 A_1$  acquires a non-zero amplitude, so that the threshold located at  $s = 4m_1^2$  becomes the lowest one also in the  $A_1 A_2$  scattering channel. Since the pole associated to  $A_4$  is located above this threshold, it can no longer remain on the real axis, which in that region is now occupied by the new cut. The position of the pole then develops an imaginary part which, according to the general requirements for unstable particles [9], is negative and brings the pole through the cut onto the unphysical region of the Riemann surface (Fig. 1.c).

## 2 Mass corrections and decay widths

The change in the position of the poles when the temperature is moved away from its critical value can be computed perturbatively in  $\tau$  within the framework of Form Factor Perturbation

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<sup>4</sup>The energy and momentum of the particles are parameterized in terms of rapidities as  $(p^0, p^1) = (m_a \cosh \theta, m_a \sinh \theta)$ . Relativistic invariant quantities as the scattering amplitudes depend on rapidity differences only.

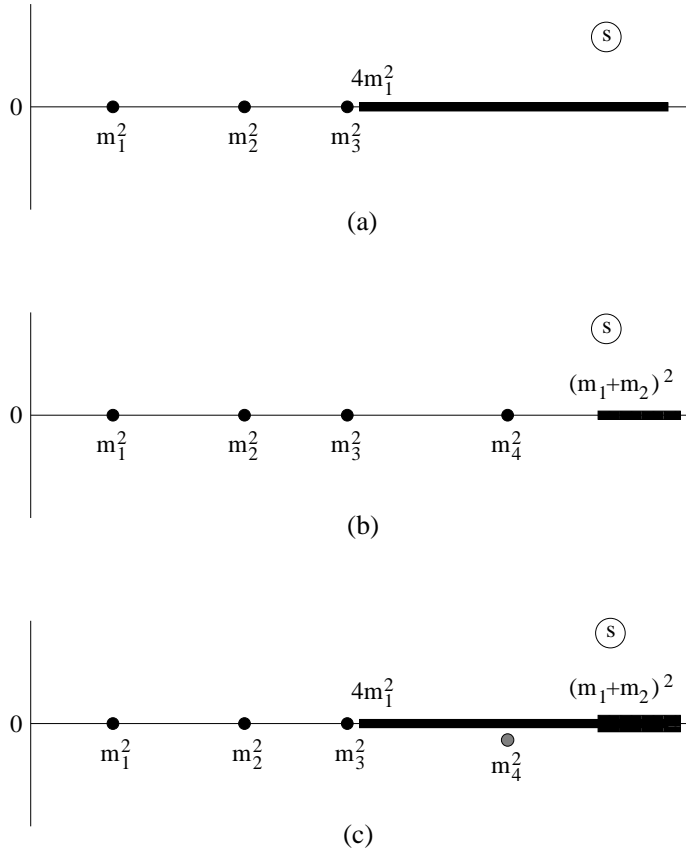


Figure 1: Poles and unitarity cuts for the scattering amplitudes  $S_{11}$  and  $S_{12}$  in the integrable case  $\tau = 0$ , (a) and (b), respectively, and for  $\tau$  slightly different from zero (c). In (c) the particle  $A_4$  became unstable and the associated pole moved through the cut into the unphysical region.

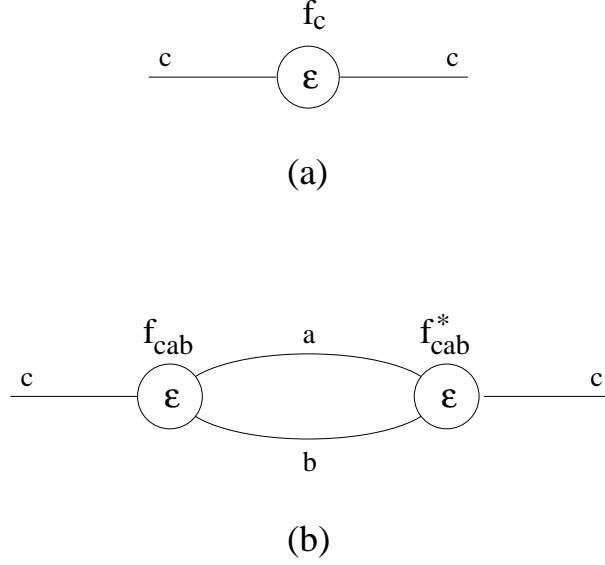


Figure 2: Diagrams determining the leading corrections to the real (a) and imaginary (b) parts of the masses at small  $\tau$ . In (b) also the intermediate particles are on shell and energy-momentum is conserved at each vertex. For  $c > 5$  also diagrams with more than two particles in the intermediate state contribute to the imaginary part.

Theory around integrable models [4]. The form factors of an operator  $\Phi(x)$  are its matrix elements between the vacuum and the multiparticle asymptotic states. We denote them as

$$F_{a_1 \dots a_n}^\Phi(\theta_1, \dots, \theta_n) = \langle 0 | \Phi(0) | A_{a_1}(\theta_1) \dots A_{a_n}(\theta_n) \rangle. \quad (10)$$

Then the leading corrections to the real and imaginary parts of the particle masses can be written as (see Fig. 2)

$$\delta \text{Re } m_c^2 \simeq 2\tau f_c, \quad (11)$$

$$\text{Im } m_c^2 = - \sum_{a \leq b, m_a + m_b \leq m_c} m_c \Gamma_{c \rightarrow ab} \simeq -\tau^2 \sum_{a \leq b, m_a + m_b \leq m_c} 2^{1-\delta_{ab}} \frac{|f_{cab}|^2}{m_c m_a \left| \sinh \theta_a^{(cab)} \right|}, \quad (12)$$

where  $\Gamma_{c \rightarrow ab}$  is the decay width of the particle  $A_c$  into the channel  $A_a A_b$  and

$$f_c = F_{cc}^\varepsilon(i\pi, 0)|_{\tau=0} \quad (13)$$

$$f_{cab} = F_{cab}^\varepsilon(i\pi, \theta_a^{(cab)}, \theta_b^{(cab)}) \Big|_{\tau=0}. \quad (14)$$

The rapidities  $\theta_a^{(cab)}$  and  $\theta_b^{(cab)}$  are determined by energy-momentum conservation for the decay  $A_c \rightarrow A_a A_b$  in the rest frame of  $A_c$ . All the masses appearing in the r.h.s. of (12) are taken at  $\tau = 0$ . For  $c > 5$  the sum in (12) must be completed including the contributions of the decay channels with more than two particles in the final state. Once the decay widths  $\Gamma_{c \rightarrow ab}$

are known, one can determine the lifetime  $t_c$  of the unstable particle  $A_c$ , given by

$$t_c = \frac{1}{\Gamma_c}, \quad \Gamma_c = \sum_{a \leq b} \Gamma_{c \rightarrow ab}. \quad (15)$$

The form factor problem for the Ising field theory in a magnetic field at  $\tau = 0$  has been studied in [10, 11], where one and two-particle form factors have been computed, and these results are reviewed in [8]. We have now extended the form factor bootstrap to the three-particle case in order to compute the decay widths (12). In the next section we recall the main steps of this approach and refer the reader to [8] for details.

### 3 Form Factors

Form factors in an integrable quantum field theory satisfy a set of equations [12, 13] which in the case of neutral particles read

$$F_{a_1 \dots a_i a_{i+1} \dots a_n}^\Phi(\theta_1, \dots, \theta_i, \theta_{i+1}, \dots, \theta_n) = S_{a_i a_{i+1}}(\theta_i - \theta_{i+1}) F_{a_1 \dots a_{i+1} a_i \dots a_n}^\Phi(\theta_1, \dots, \theta_{i+1}, \theta_i, \dots, \theta_n) \quad (16)$$

$$F_{a_1 \dots a_n}^\Phi(\theta_1 + 2i\pi, \theta_2, \dots, \theta_n) = F_{a_2 \dots a_n a_1}^\Phi(\theta_2, \dots, \theta_n, \theta_1) \quad (17)$$

$$\text{Res}_{\theta_a - \theta_b = iu_{ab}^c} F_{aba_1 \dots a_n}^\Phi(\theta_a, \theta_b, \theta_1, \dots, \theta_n) = ig_{ab}^c F_{ca_1 \dots a_n}^\Phi(\theta_c, \theta_1, \dots, \theta_n) \quad (18)$$

$$\begin{aligned} \text{Res}_{\theta' = \theta + i\pi} F_{aba_1 \dots a_n}^\Phi(\theta', \theta, \theta_1, \dots, \theta_n) = \\ i\delta_{ab} \left( 1 - \prod_{j=1}^n S_{a_j a}(\theta_j - \theta) \right) F_{a_1 \dots a_n}^\Phi(\theta_1, \dots, \theta_n), \end{aligned} \quad (19)$$

with the three-particle couplings  $g_{ab}^c$  entering (18) determined from the scattering amplitudes through the relations

$$S_{ab}(\theta \simeq iu_{ab}^c) \simeq \frac{i(g_{ab}^c)^2}{\theta - iu_{ab}^c}. \quad (20)$$

A parameterization of the form factors which solves (16) and 17) and contains the poles prescribed by (18) and (19) is given by

$$F_{a_1 \dots a_n}^\Phi(\theta_1, \dots, \theta_n) = Q_{a_1 \dots a_n}^\Phi(\theta_1, \dots, \theta_n) \prod_{i < j} \frac{F_{a_i a_j}^{min}(\theta_i - \theta_j)}{(e^{\theta_i} + e^{\theta_j})^{\delta_{a_i a_j}} D_{a_i a_j}(\theta_i - \theta_j)}. \quad (21)$$

Here

$$F_{ab}^{min}(\theta) = \left( -i \sinh \frac{\theta}{2} \right)^{\delta_{ab}} \prod_{\gamma \in \mathcal{G}_{ab}} (T_{\gamma/30}(\theta))^{p_\gamma}, \quad (22)$$

with

$$T_\alpha(\theta) = \exp \left\{ 2 \int_0^\infty \frac{dt}{t} \frac{\cosh(\alpha - \frac{1}{2})t}{\cosh \frac{t}{2} \sinh t} \sin^2 \frac{(i\pi - \theta)t}{2\pi} \right\} \quad (23)$$

satisfying

$$T_\alpha(\theta) = -t_\alpha(\theta) T_\alpha(-\theta) \quad (24)$$

$$T_\alpha(\theta + 2i\pi) = T_\alpha(-\theta) \quad (25)$$

$$T_\alpha(\theta) \sim \exp |\theta|/2, \quad |\theta| \rightarrow \infty. \quad (26)$$

The quantities  $\gamma$  and  $p_\gamma$  in (22) coincide with those entering the expression

$$S_{ab}(\theta) = \prod_{\gamma \in \mathcal{G}_{ab}} (t_{\gamma/30}(\theta))^{p_\gamma} \quad (27)$$

for the amplitudes ( $S_{ab}(0) = (-1)^{\delta_{ab}}$ ). The dynamical poles are inserted in (21) through the factors

$$D_{ab}(\theta) = \prod_{\gamma \in \mathcal{G}_{ab}} (\mathcal{P}_{\gamma/30}(\theta))^{i_\gamma} (\mathcal{P}_{1-\gamma/30}(\theta))^{j_\gamma}, \quad (28)$$

where

$$\mathcal{P}_\alpha(\theta) = \frac{\cos \pi\alpha - \cosh \theta}{2 \cos^2 \frac{\pi\alpha}{2}} \quad (29)$$

and

$$\begin{aligned} i_\gamma &= n+1, & j_\gamma &= n, & \text{if } p_\gamma &= 2n+1 \\ i_\gamma &= n, & j_\gamma &= n, & \text{if } p_\gamma &= 2n. \end{aligned} \quad (30)$$

Multiple poles corresponding to  $p_\gamma > 1$  are related to multiscattering processes [14] and give rise to residue equations which generalize (18) [10].

The operator dependence in (21) is contained in the entire functions  $Q_{a_1 \dots a_n}^\Phi(\theta_1, \dots, \theta_n)$ . For scalar operators  $\Phi$  with scaling dimension  $X_\Phi$ , the asymptotic bound [10]

$$\lim_{|\theta_i| \rightarrow \infty} F_{a_1 \dots a_n}^\Phi(\theta_1, \dots, \theta_n) \sim \exp(Y_\Phi |\theta_i|) \quad (31)$$

$$Y_\Phi \leq \frac{X_\Phi}{2}, \quad (32)$$

together with the invariance under  $\theta_j \rightarrow \theta_j + 2i\pi$ , implies that the  $Q_{a_1 \dots a_n}^\Phi$  are rational functions of the variables  $x_j \equiv e^{\theta_j}$ . In the two-particle case, taking into account Lorentz invariance, the vanishing of the residue (18) and the identity  $F_{a_1 a_2}(\theta_1, \theta_2) = F_{a_2 a_1}(\theta_1, \theta_2)$ , one can write

$$Q_{a_1 a_2}^\Phi(\theta_1, \theta_2) = (x_1 + x_2)^{\delta_{a_1 a_2}} P_{a_1 a_2}^\Phi(\theta_1, \theta_2) \quad (33)$$

with

$$P_{a_1 a_2}^\Phi(\theta_1, \theta_2) = \sum_{k=0}^{N_{a_1 a_2}} c_{a_1 a_2}^{\Phi, k} \cosh^k(\theta_1 - \theta_2). \quad (34)$$

In the three-particle case we use the notation

$$\mathcal{Q}_{a_1 a_2 a_3}^\Phi(\theta_1, \theta_2, \theta_3) = \sum_{\alpha_1, \alpha_2, \alpha_3 = \alpha'_1, \alpha'_2, \alpha'_3}^{\alpha''_1, \alpha''_2, \alpha''_3} d_{a_1 a_2 a_3}^{\Phi, \{\alpha_1, \alpha_2, \alpha_3\}} x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3}. \quad (35)$$



The coefficients in (34) and (35) are real and restricted in number by the bound (32). In particular, defining  $\text{Deg}_{a_i a_j}$  through

$$\frac{F_{a_i a_j}^{\min}(\theta_i - \theta_j)}{D_{a_i a_j}(\theta_i - \theta_j)} \sim e^{\mp \text{Deg}_{a_i a_j} \theta_i}, \quad \theta_i \rightarrow \pm\infty \quad (36)$$

and

$$\text{Deg}_{i,+} = -\text{Deg}_{a_i a_j} - \text{Deg}_{a_i a_k} - \delta_{a_i a_j} - \delta_{a_i a_k} \quad (37)$$

$$\text{Deg}_{i,-} = \text{Deg}_{a_i a_j} + \text{Deg}_{a_i a_k} \quad (38)$$

( $i, j, k$  a permutation of 1, 2, 3), we conclude

$$\alpha'_i = -\text{Deg}_{i,-} \quad (39)$$

$$\alpha''_i = -\text{Deg}_{i,+} \quad (40)$$

for both  $\Phi = \sigma, \varepsilon$  in (35). Further constraints on (35) come from symmetry under permutations of identical particles

$$d_{a_i a_i a_j}^{\Phi, \{\alpha_1, \alpha_2, \alpha_3\}} = d_{a_i a_i a_j}^{\Phi, \{\alpha_2, \alpha_1, \alpha_3\}}, \quad (41)$$

and from Lorentz invariance, which for scalar operators implies

$$\sum_{k=1}^3 \alpha_k = \sum_{1 \leq i < j \leq 3} \delta_{a_i a_j}. \quad (42)$$

The form factor equations together with the asymptotic bound allow to show that the space of non-trivial relevant scalar operators is two-dimensional [8], as expected for the Ising model, but cannot resolve the linear combination between  $\sigma$  and  $\varepsilon$ . This is achieved imposing the asymptotic factorization condition

$$\lim_{\alpha \rightarrow +\infty} F_{a_1 \dots a_r b_1 \dots b_l}^{\Phi}(\theta_1 + \alpha, \dots, \theta_r + \alpha, \theta'_1, \dots, \theta'_l) = \frac{1}{\langle \Phi \rangle} F_{a_1 \dots a_r}^{\Phi}(\theta_1, \dots, \theta_r) F_{b_1 \dots b_l}^{\Phi}(\theta'_1, \dots, \theta'_l), \quad (43)$$

which holds, in particular, for relevant scalar scaling operators in theories without internal symmetries [15]. Of the two form factor solutions obtained in this way, that corresponding to  $\sigma$  is picked up exploiting the proportionality between this operator and the trace of the energy-momentum tensor at  $\tau = 0$  [10, 11]; the remaining solution must then correspond to  $\varepsilon$ . Once the initial conditions of the form factor bootstrap for  $\Phi = \sigma, \varepsilon$  have been fixed in this way, the functions  $Q_{a_1 \dots a_n}^{\Phi}$  in (21) for these operators can be uniquely determined using the residue equations. We give in [16] a list of results for  $n = 2$  (which extends that given in [10, 11]) and for  $n = 3, 4$ .

## 4 Results

From the above results we extract the following values to be used in (11) and (12)

$$f_1 = (-17.8933..) \langle \varepsilon \rangle_{\tau=0} \quad (44)$$

$$f_2 = (-24.9467..) \langle \varepsilon \rangle_{\tau=0} \quad (45)$$

$$f_3 = (-53.6799..) \langle \varepsilon \rangle_{\tau=0} \quad (46)$$

$$f_4 = (-49.3206..) \langle \varepsilon \rangle_{\tau=0} \quad (47)$$

$$|f_{411}| = (36.73044..) |\langle \varepsilon \rangle|_{\tau=0} \quad (48)$$

$$|f_{511}| = (19.16275..) |\langle \varepsilon \rangle|_{\tau=0} \quad (49)$$

$$|f_{512}| = (11.2183..) |\langle \varepsilon \rangle|_{\tau=0} . \quad (50)$$

Within the normalization (5) the vacuum expectation value appearing in these results takes the value [17]

$$\langle \varepsilon \rangle_{\tau=0} = (2.00314..) |h|^{8/15} . \quad (51)$$

The variation of the mass ratios

$$r_a = \frac{\text{Re } m_a}{m_1} , \quad (52)$$

close to the magnetic axis, is given by

$$\delta r_a = -\frac{\tau f_1}{m_1 m_a} \left( r_a^2 - \frac{f_a}{f_1} \right) + O(\tau^2) . \quad (53)$$

It can be immediately checked that the values (44)-(47) imply positive coefficients for  $\tau$  in this equation when  $a = 1, \dots, 4$ , in agreement with the expectation that the distance of the stable (unstable) particles from the threshold  $2m_1$  decreases (increases) as  $\eta$  increases. The mass variations of the first three particles close to the magnetic axis were measured in [4, 18] and fully agree with the predictions of the Form Factor Perturbation Theory.

It is tempting to ignore the non-linear corrections to (53) and obtain a first-order estimate  $\eta_n^{(1)}$  of the values  $\eta_n$  at which the  $n$ -th particle decays. These are determined by the condition

$$r_n(\eta_n) = 2 , \quad n > 1 , \quad (54)$$

and, in this way, one obtains

$$\eta_2^{(1)} \simeq 0.2 \quad (55)$$

$$\eta_3^{(1)} \simeq 0.01 \quad (56)$$

$$\eta_4^{(1)} \simeq -0.2 . \quad (57)$$

Comparison with (4) shows that the first-order estimate captures the correct order of magnitude of the decay point for  $n = 2, 3$ . Notice that, if for  $n = 3$  the mass ratio changes of only 0.5% from  $\eta = 0$  to  $\eta_3$ , the variation of  $m_3$  and  $m_1$  with respect to  $m_1|_{\eta=0}$  in the linear approximation

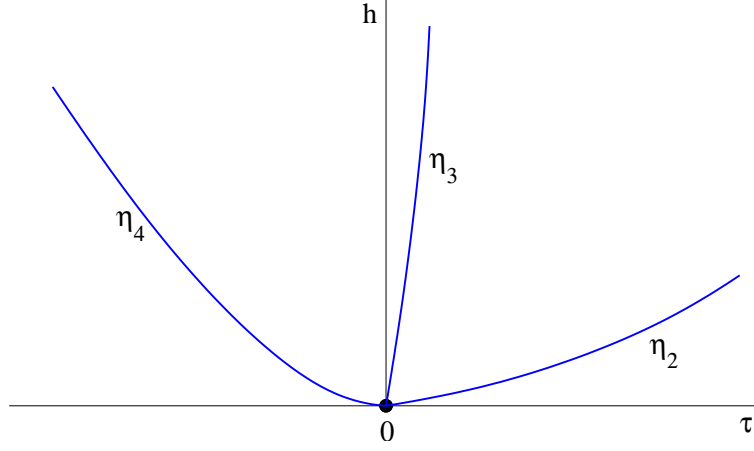


Figure 3: The first few trajectories which divide the  $h$ - $\tau$  plane into regions with a different number of stable particles. There are  $n$  stable particles in between the trajectories labeled by  $\eta_n$  and  $\eta_{n+1}$ . These trajectories densely fill the plane when the negative horizontal axis is approached ( $\eta \rightarrow -\infty$ ).

is considerably larger. The trajectories corresponding to  $\eta_2$ ,  $\eta_3$  and to the first-order estimate for  $\eta_4$  are shown in Fig. 3.

For the imaginary parts of the first two particles above threshold we obtain

$$\text{Im } m_4^2 \simeq (-840.172..) \left( \frac{\tau \langle \varepsilon \rangle_{\tau=0}}{m_1} \right)^2 = (-173.747..) \tau^2 \quad (58)$$

$$\text{Im } m_5^2 \simeq (-240.918..) \left( \frac{\tau \langle \varepsilon \rangle_{\tau=0}}{m_1} \right)^2 = (-49.8217..) \tau^2 . \quad (59)$$

The ratio of lifetimes

$$\lim_{\tau \rightarrow 0} \frac{t_4}{t_5} = \lim_{\tau \rightarrow 0} \frac{m_4 \text{Im } m_5^2}{m_5 \text{Im } m_4^2} = 0.23326.. \quad (60)$$

is universal. While  $A_4$  can only decay into  $A_1 A_1$ ,  $A_5$  has also the  $A_1 A_2$  channel available. The relevant branching ratios

$$b_{c \rightarrow ab} = \frac{m_c|_{\tau=0} \Gamma_{c \rightarrow ab}}{|\text{Im } m_c^2|} \quad (61)$$

are

$$\lim_{\tau \rightarrow 0} b_{5 \rightarrow 11} = 0.47364.., \quad \lim_{\tau \rightarrow 0} b_{5 \rightarrow 12} = 0.52635.. \quad (62)$$

It appears from (60) that the lifetime of  $A_5$  is more than four times longer than that of  $A_4$ , and this seems to contradict, somehow, the expectation inherited from accelerator physics that the lifetime of an unstable particle decreases as its mass increases. Notice, however, that the  $d$ -dimensional phase space for the decay  $A_c \rightarrow A_a A_b$  is

$$\int \frac{d^{d-1} \vec{p}_a}{p_a^0} \frac{d^{d-1} \vec{p}_b}{p_b^0} \delta^d(p_a + p_b - p_c) \sim \frac{p^{d-3}}{m_c}, \quad (63)$$

where  $p = |\vec{p}_a| = |\vec{p}_b|$  is taken in the center of mass frame. For fixed decay products,  $p$  increases with  $m_c$  and, in  $d = 2$ , it joins the factor  $m_c$  in the denominator, leading therefore to a suppression of the phase space. The results (48)–(50) show that such a suppression of the decay width is further enhanced by the dynamics in a way that is not compensated by the opening of additional decay channels.

In [16] we also give the four-particle form factor  $F_{1111}^\varepsilon|_{\tau=0}$  which determines the first-order variation to the elastic scattering amplitude  $S_{11}$  through the formula [4]

$$\delta S_{11}(\theta_1 - \theta_2) \simeq -\frac{i\tau}{m_1^2 \sinh(\theta_1 - \theta_2)} \lim_{\delta \rightarrow 0} F_{1111}^\varepsilon(\theta_1 + i\pi + \delta, \theta_2 + i\pi + \delta, \theta_1, \theta_2) . \quad (64)$$

Here the limit is needed to deal with the singularities prescribed by (19).

## 5 Numerical studies

Checks of the analytic results for the particle spectrum may come, in particular, from the numerical diagonalization of the Hamiltonian on a cylinder geometry, the results for the plane being recovered as the circumference  $R$  of the cylinder goes to infinity. For the stable particles of the Ising model in a magnetic field this kind of analysis has been performed both in the continuum [4, 5, 19] and on the lattice [18, 20, 21], always confirming the analytic predictions. The numerical tests are more complicated for unstable particles.

Qualitatively, the signature of unstable particles on the cylinder is very clear [22]. At the integrable point, when the energy levels are plotted as a function of  $R$ , the line corresponding to a particle above threshold crosses infinitely many levels which belong to the continuum when  $R = \infty$  (Fig. 4). Once integrability is broken, this line “disappears” through a removal of level crossings and a reshaping of the lines associated to stable excitations (Fig. 5). Quantitatively, though, it is not so obvious how the decay width can be measured from such a spectrum. In absence of a direct measurement of this quantity, one could try however to estimate, using the exact matrix elements computed above, the energy splitting resulting from the removal of a level crossing taking place at a sufficiently large value  $R^*$  of the cylinder circumference.

The usual formula for the first order energy splitting of two states  $|1\rangle$  and  $|2\rangle$ , which are degenerate in the unperturbed system, is given by

$$\Delta E = \sqrt{(V_{11} - V_{22})^2 + 4|V_{12}|^2}, \quad (65)$$

where  $V_{ij} = \langle i|V|j\rangle$  are the matrix elements of the perturbing Hamiltonian  $V$ . When we put the theory (1) on the cylinder and perturb around  $\tau = 0$ , we have

$$V = -\tau \int_0^R dx \, \varepsilon(x) . \quad (66)$$

For  $R^*$  large enough, the corrections coming from the finite volume dynamics can be neglected in the first approximation. Hence, the only  $R$  dependence comes from the change of normalization of the particle states when passing from the plane to the cylinder:

$$\langle A_b(p_b)|A_a(p_a)\rangle = 2\pi\delta_{ab}p_a^0\delta(p_a^1 - p_b^1) \longrightarrow \delta_{ab}\delta_{p_a^1 p_b^1}p_a^0 R . \quad (67)$$

By taking into account the normalization  $\langle i|j \rangle = \delta_{ij}$  of the states entering (65), the matrix elements to be used for the removal of the crossing between the levels corresponding to  $A_4$  and  $A_1 A_1$  are then given by

$$V_{11} = -\tau R^* \frac{f_4}{m_4 R^*} , \quad (68)$$

$$V_{22} = -\tau R^* \frac{f_{1111}}{(\frac{m_4}{2} R^*)^2} , \quad (69)$$

$$V_{12} = -\tau R^* \frac{f_{411}}{\frac{1}{2}(m_4 R^*)^{3/2}} , \quad (70)$$

where

$$\begin{aligned} f_{1111} &= \lim_{\delta \rightarrow 0} F_{1111}^\varepsilon(\theta_1^{(411)} + i\pi + \delta, -\theta_1^{(411)} + i\pi + \delta, -\theta_1^{(411)}, \theta_1^{(411)}) \Big|_{\tau=0} \\ &= (408.78..) \langle \varepsilon \rangle_{\tau=0} . \end{aligned} \quad (71)$$

For the purpose of comparison with numerical data it can be useful to consider the universal ratio  $\Delta E / \delta E_{gs}(R^*)$ , where

$$\delta E_{gs}(R) = -R \tau \langle \varepsilon \rangle_{\tau=0} \quad (72)$$

is the first order variation of the ground state energy.

Energy spectra on the cylinder can be obtained directly in the continuum through the truncated conformal space approach [23], in which the Hamiltonian is diagonalized numerically on a finite dimensional subspace of the conformal states relative to the critical point. This is the method that we have used to obtain the numerical spectrum. By including states up to the 5th level of the Verma module of the conformal families of the Ising model, our final Hamiltonian has been truncated up to 43 states. Even though this truncation of the Hilbert space has proved to be successful to measure mass ratio corrections and the change of the ground state energy, it fails however to measure with sufficient precision the energy splittings which take place above threshold at large values of  $R$ . This was not totally unexpected since it is well known that the accuracy in the determination of the hamiltonian levels decreases as  $R$  and/or the energy increase, due to the truncation effects. The remedy to this problem consists, of course, in using a larger Hamiltonian, in order to reach a sufficient level of precision in the region of large  $R$  where we want to measure the energy splittings between the higher mass line and the threshold lines. In this respect, it would be interesting to make such a check by using the method employed in [5], where the diagonalization is performed on a larger number of states of the free fermionic basis corresponding to  $h = 0$ .

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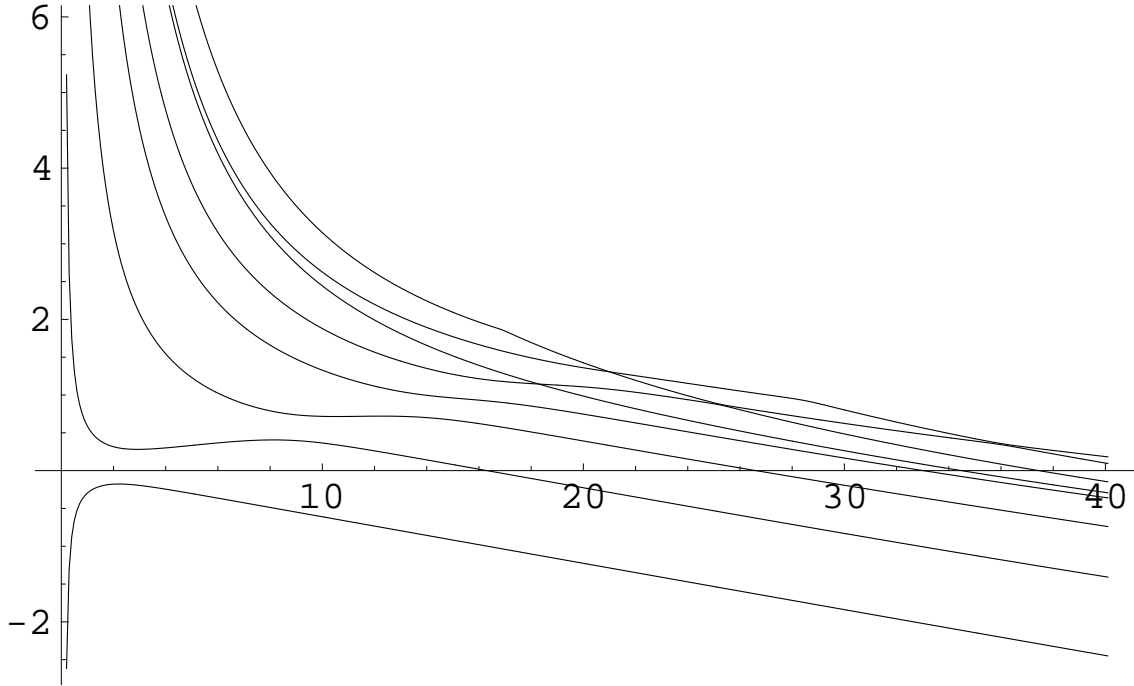


Figure 4: First eight energy levels of the finite volume Hamiltonian of the scaling Ising model with magnetic field at critical temperature as functions of the scaling variable  $r = m_1 R$ . At  $r = 40$ , starting from the bottom, the levels are identified as the ground state, the first three particle states  $A_1$ ,  $A_2$  and  $A_3$ , three scattering states  $A_1 A_1$ , the particle above threshold  $A_4$ . Crossings between the line associated to the latter and the scattering states are visible around  $r = 18$ ,  $r = 25$  and  $r = 36$ .

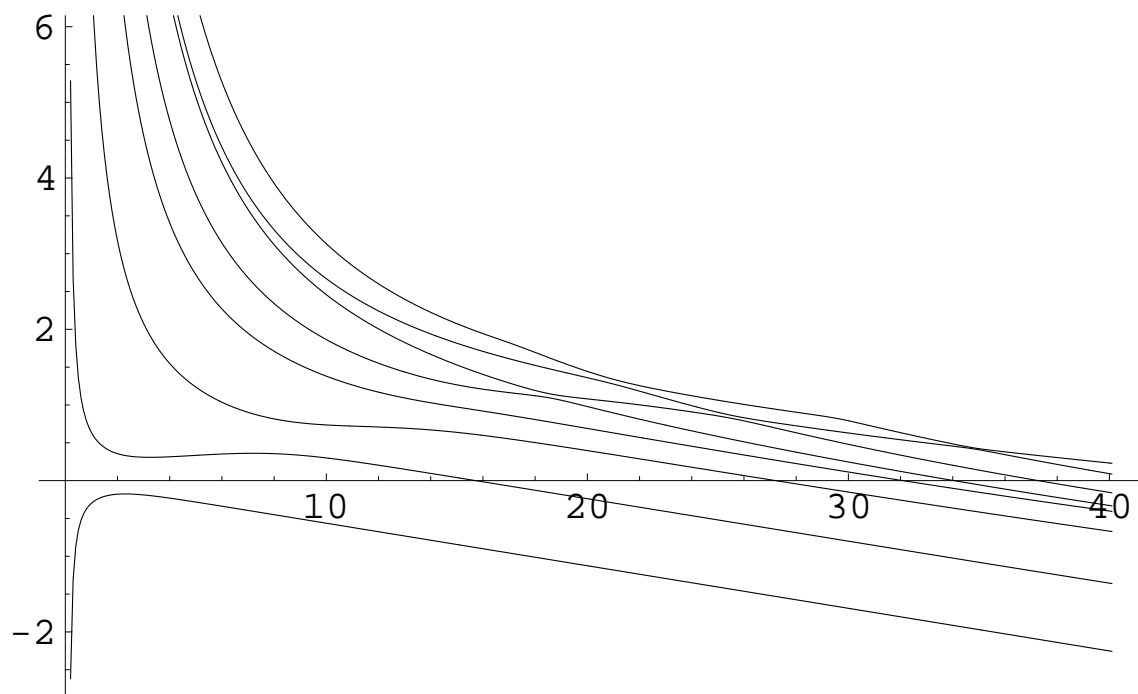


Figure 5: First eight energy levels of the finite volume Hamiltonian of the scaling Ising model with magnetic field slightly away from the critical temperature. Observe the splitting of the crossings pointed out in the previous figure.